Level-One Representations and Vertex Operators of Quantum Affine Superalgebra $U_q[gl(\widehat{N|N})]$

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Abstract

Level-one representations of the quantum affine superalgebra $U_q[gl(\widehat{N|N})]$ associated to the appropriate non-standard system of simple roots and q-vertex operators (intertwining operators) associated with the level-one modules are constructed explicitly in terms of free bosonic fields.

I   Introduction

The algebraic analysis approach [1, 2] based on quantum affine algebra symmetries enables one not only to solve massive or off-critical integrable models directly in the thermodynamic limit but also to compute their correlation functions [3] and form factors [4] in the form of integrals by applying the techniques similar to those used so successfully in the critical cases (see, c.f. [5]). The key components behind this method are infinite dimensional highest weight representations of the quantum affine algebras and the corresponding q-vertex operators [6] which are intertwiners of these representations. As in the critical cases, this procedure requires the explicit construction of the highest weight representations and vertex operators in terms of free bosonic fields.

By now, the level-one representations and vertex operators have been constructed in terms of free bosons for most quantum affine bosonic algebras (see, e.g. [7, 8, 9, 10, 11, 12]). In contrast, much less has been known for the case of quantum affine superalgebras. For the type I quantum affine superalgebra $U_q[gl(\widehat{M|N})]$, $M \neq N$, the level-one representations and vertex operators have been investigated in [13] (see [14] for a level-k free boson realization of $U_q[sl(\widehat{2|1})]$). In particular, the level-one irreducible highest weight representations of $U_q[gl(\widehat{2|1})]$ were studied in some details and the correpsonding characters were derived [13]. These representations have been re-examined and used to compute the correlation functions of the q-deformed supersymmetric t-J model in [15].

So far in the literature, the very interesting case of $M = N$ has been largely ignored. The only exception is [16] where the special case of $M = N = 2$ was treated and the type I vertex operators involving infinite dimensional evaluation (or level-zero) representations were also constructed for this special case. By contrast, we shall consider the general $M = N$ case and investigate both type I and type II vertex operators with respect to finite dimensional evaluation modules. The $M = N$ case is interesting since it seems to us that $U_q[gl(\widehat{N|N})]$ is the only untwisted superalgebra which has a non-standard system where all simple roots are odd or fermionic. It also seems to be the only superalgebra where a vertex type quasi-Hopf twistor can be constructed [17] and thus the correpsonding elliptic quantum supergroup $\mathcal{A}_{q,p}[gl(\widehat{N|N})]$ can be introduced.

In this paper, we construct a level-one representation of $U_q[gl(\widehat{N|N})]$ by bosonizing the Drinfeld generators. We also construct the vertex operators associated with the level-one representations in terms of the free bosonic fields.

The layout of this paper is the following. In section 2, we describe the Drinfeld realization [18] of $U_q[gl(\widehat{N|N})]$ in the non-standard system of simple roots and determine

the "main terms" [19] in the coproduct formulae of the Drinfeld generators. In section 3, we derive the $2N$-dimensional evaluation (or level-zero) representations of $U_q[gl(\widehat{N|N})]$. In section 4, we investigate the bosonization of $U_q[gl(\widehat{N|N})]$ and construct an explicit level-one representation in terms of free bosonic fields. Section 5 is devoted to the study of the bosonization of the level-one vertex operators.

II Quantum Affine Superalgebra $U_q[gl(\widehat{N|N})]$

As is well-known, a given Kac-Moody superalgebra [20] allows many inequivalent systems of simple roots. A system of simple roots is called distinguished if it has minimal odd roots. Let $\{\alpha_i, \ i = 0, 1, \cdots, 2N - 1\}$ denote a chosen set of simple roots of the affine superalgebra $gl(\widehat{N|N})$. Let (,) be a fixed invariant bilinear form on the root space. Let $\mathcal{H}$ be the Cartan subalgebra and throughout we identify the dual $\mathcal{H}^*$ with $\mathcal{H}$ via (,). As is shown in [17], $gl(\widehat{N|N})$ has a simple root system in which all simple roots are odd (or fermionic). This system can be constructed from the distinguished simple root system by using the "extended" Weyl operation [21] repeatedly. We have the following simple roots, all of which are odd (or fermionic)

$$
\begin{aligned}
\alpha_0 &= \delta - \varepsilon_1 + \varepsilon_{2N}\,, \\
\alpha_l &= \varepsilon_l - \varepsilon_{l+1}\,, \qquad l = 1, 2, \cdots, 2N - 1
\end{aligned}
\tag{II.1}
$$

with δ, $\{\varepsilon_k\}_{k=1}^{2N}$ satisfying

$$
(\delta, \delta) = (\delta, \varepsilon_k) = 0, \qquad (\varepsilon_k, \varepsilon_{k'}) = (-1)^{k+1}\delta_{kk'}.
\tag{II.2}
$$

Such a simple root system is usually called non-standard. The generalized symmetric Cartan matrix $(a_{ii'})$ takes the form

$$
\begin{aligned}
a_{01} &\equiv (\alpha_0, \alpha_1) = -1, \qquad a_{0,2N-1} \equiv (\alpha_0, \alpha_{2N-1}) = 1, \\
a_{ll'} &\equiv (\alpha_l, \alpha_{l'}) = (-1)^{l+1}(\delta_{l,l'-1} - \delta_{l,l'+1}), \quad l, l' = 1, 2, \cdots, 2N - 1.
\end{aligned}
\tag{II.3}
$$

This Cartan matrix is degenrate. To obtain a non-degenerate Cartan matrix, we extend [22] $\mathcal{H}$ by adding to it the element

$$
\alpha_{2N} = \sum_{k=1}^{2N} \varepsilon_k.
\tag{II.4}
$$

In the following, we denote by $\tilde{\mathcal{H}}$ the extended Cartan subalgebra and by $\tilde{\mathcal{H}}^*$ the dual of $\tilde{\mathcal{H}}$. The enlarged Cartan matrix has the following extra matrix elements:

$$
a_{2N,2N} \equiv (\alpha_{2N}, \alpha_{2N}) = 0, \qquad a_{i,2N} \equiv (\alpha_i, \alpha_{2N}) = 2 \cdot (-1)^{i+1}.
\tag{II.5}
$$

Let $\{h_0, h_1, \cdots, h_{2N}, d\}$ be a basis of $\tilde{\mathcal{H}}$, where h_{2N} is the element in $\tilde{\mathcal{H}}$ corresponding to α_{2N} and d is the usual derivation operator. We shall write $h_i = \alpha_i$ $(i = 0, 1, \cdots, 2N)$ with α_i given by (II.1, II.4). Let $\{\Lambda_0, \Lambda_1, \cdots, \Lambda_{2N}, c\}$ be the dual basis with Λ_j being fundamental weights and c the canonical central element. We have [17]

$$\Lambda_{2N} = \frac{1}{2N} \sum_{k=1}^{2N} (-1)^{k+1} \varepsilon_k,$$

$$\Lambda_i = d + \sum_{k=1}^{i} (-1)^{k+1} \varepsilon_k - \frac{i}{2N} \sum_{k=1}^{2N} (-1)^{k+1} \varepsilon_k, \tag{II.6}$$

where $i = 0, 1, \cdots, 2N - 1$.

The quantum affine superalgebra $U_q[gl(\widehat{N|N})]$ is a quantum (or q-) deformation of the universal enveloping algebra of $gl(\widehat{N|N})$ and is generated by the Chevalley generators $\{e_i, \ f_i \ q^{h_j}, \ d | i = 0, 1, \cdots, 2N - 1, \ j = 0, 1, \cdots, 2N\}$. The $\mathbf{Z}_2$-grading of the Chevalley generators is $[e_i] = [f_i] = 1$, $i = 0, 1, \cdots, 2N - 1$ and zero otherwise. The defining relations are

$$hh' = h'h, \qquad \forall h \in \tilde{\mathcal{H}},$$

$$q^{h_j} e_i q^{-h_j} = q^{a_{ij}} e_i, \qquad [d, e_i] = \delta_{i0} e_i,$$

$$q^{h_j} f_i q^{-h_j} = q^{-a_{ij}} f_i, \qquad [d, f_i] = -\delta_{i0} f_i,$$

$$[e_i, f_{i'}] = \delta_{ii'} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

$$[e_i, e_{i'}] = [f_i, f_{i'}] = 0, \qquad \text{for} \quad a_{ii'} = 0,$$

$$[[e_0, e_1]_{q^{-1}}, [e_0, e_{2N-1}]_q] = 0,$$

$$[[e_l, e_{l-1}]_{q^{(-1)^l}}, [e_l, e_{l+1}]_{q^{(-1)^{l+1}}}] = 0,$$

$$[[e_{2N-1}, e_{2N-2}]_{q^{-1}}, [e_{2N-1}, e_0]_q] = 0,$$

$$[[f_0, f_1]_{q^{-1}}, [f_0, f_{2N-1}]_q] = 0,$$

$$[[f_l, f_{l-1}]_{q^{(-1)^l}}, [f_l, f_{l+1}]_{q^{(-1)^{l+1}}}] = 0,$$

$$[[f_{2N-1}, f_{2N-2}]_{q^{-1}}, [f_{2N-1}, f_0]_q] = 0, \qquad l = 1, 2, \cdots, 2N - 2. \tag{II.7}$$

Here and throughout, $[a, b]_x \equiv ab - (-1)^{[a][b]} x ba$ and $[a, b] \equiv [a, b]_1$. The four-th order q-Serre relations are obtained by using Yamane's Dynkin diagram procedure [23].

$U_q[gl(\widehat{N|N})]$ is a $\mathbf{Z}_2$-graded quasi-triangular Hopf algebra endowed with the following coproduct Δ, counit ϵ and antipode S:

$$\begin{aligned}
\Delta(h) &= h \otimes 1 + 1 \otimes h, \\
\Delta(e_i) &= e_i \otimes 1 + q^{h_i} \otimes e_i, \qquad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \\
\epsilon(e_i) &= \epsilon(f_i) = \epsilon(h) = 0, \\
S(e_i) &= -q^{-h_i} e_i, \qquad S(f_i) = -f_i q^{h_i}, \qquad S(h) = -h, \tag{II.8}
\end{aligned}$$

where $i = 0, 1, \cdots, 2N - 1$ and $h \in \tilde{\mathcal{H}}$. Notice that the antipode S is a $\mathbf{Z}_2$-graded algebra anti-homomorphism. Namely, for any homogeneous elements $a, b \in U_q[gl(\widehat{N|N})]$ $S(ab) = (-1)^{[a][b]} S(b) S(a)$, which extends to inhomogeneous elements through linearity. Moreover,

$$S^2(a) = q^{-2\rho} \, a \, q^{2\rho}, \qquad \forall a \in U_q[gl(\widehat{N|N})], \tag{II.9}$$

where ρ is an element in $\tilde{\mathcal{H}}$ such that $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$ for any simple root α_i, $i = 0, 1, 2, \cdots, 2N - 1$. Explicitly,

$$\rho = \frac{1}{2} \sum_{k=1}^{2N} (-1)^k \varepsilon_k, \tag{II.10}$$

which coincides with $\bar{\rho}$, the half-sum of positive roots of $gl(N|N)$ in the present simple root system. The multiplication rule on the tensor products is $\mathbf{Z}_2$-graded: $(a \otimes b)(a' \otimes b') = (-1)^{[b][a']}(aa' \otimes bb')$ for any homogeneous elements $a, b, a', b' \in U_q[gl(\widehat{N|N})]$. We also introduce the element in $\tilde{\mathcal{H}}$

$$\tilde{\rho} = \sum_{i=0}^{2N-1} \Lambda_i + \xi N \Lambda_{2N}, \tag{II.11}$$

which gives the principal gradation

$$[\tilde{\rho}, e_i] = e_i, \qquad [\tilde{\rho}, f_i] = -f_i, \qquad i = 0, 1, \cdots, 2N - 1. \tag{II.12}$$

In (II.11), ξ is an arbitrary constant.

$U_q[gl(\widehat{N|N})]$ can also be realized in terms of the Drinfeld generators [18] $\{X_m^{\pm,i}, \ H_n^j, \ q^{\pm H_0^j}, \ c, \ d | m \in \mathbf{Z}, \ n \in \mathbf{Z} - \{0\}, \ i = 1, 2, \cdots, 2N - 1, \ j = 1, 2, \cdots, 2N\}$. The $\mathbf{Z}_2$-grading of the Drinfeld generators is given by $[X_m^{\pm,i}] = 1$ for all $i = 1, \cdots, 2N - 1$, $m \in \mathbf{Z}$ and $[H_n^j] = [H_0^j] = [c] = [d] = 0$ for all $j = 1, \cdots, 2N$, $n \in \mathbf{Z} - \{0\}$. The relations satisfied by the Drinfeld generators read (see [23, 24] for the Drinfeld realization of $U_q[gl(\widehat{N|N})]$ in the distinguished system of simple roots)

$$[c, a] = [d, H_0^j] = [H_0^j, H_n^{j'}] = 0, \qquad \forall a \in U_q[gl(\widehat{N|N})]$$

$$q^{H_0^j} X_n^{\pm,i} q^{-H_0^j} = q^{\pm a_{ij}} X_n^{\pm,i},$$

$$[d, X_n^{\pm,i}] = n X_n^{\pm,i}, \quad [d, H_n^j] = n H_n^j,$$

$$[H_n^j, H_m^{j'}] = \delta_{n+m,0} \frac{[a_{jj'} n]_q [nc]_q}{n},$$

$$[H_n^j, X_m^{\pm,i}] = \pm \frac{[a_{ij} n]_q}{n} X_{n+m}^{\pm,i} q^{\mp |n| c/2},$$

$$[X_n^{+,i}, X_m^{-,i'}] = \frac{\delta_{ii'}}{q - q^{-1}} \left(q^{\frac{c}{2}(n-m)} \psi_{n+m}^{+,i} - q^{-\frac{c}{2}(n-m)} \psi_{n+m}^{-,i} \right),$$

$$[X_n^{\pm,i}, X_m^{\pm,i'}] = 0, \qquad \text{for } a_{ii'} = 0,$$

$$[X_{n+1}^{\pm,i}, X_m^{\pm,i'}]_{q^{\pm a_{ii'}}} - [X_{m+1}^{\pm,i'}, X_n^{\pm,i}]_{q^{\pm a_{ii'}}} = 0,$$

$$[[X_m^{\pm,l}, X_{m'}^{\pm,l-1}]_{q^{(-1)^l}}, [X_n^{\pm,l}, X_{n'}^{\pm,l+1}]_{q^{(-1)^{l+1}}}]$$

$$+ [[X_n^{\pm,l}, X_{m'}^{\pm,l-1}]_{q^{(-1)^l}}, [X_m^{\pm,l}, X_{n'}^{\pm,l+1}]_{q^{(-1)^{l+1}}}] = 0, \quad l = 2, \cdots, 2N-2. \text{ (II.13)}$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ and $\psi_n^{\pm,j}$ are related to $H_{\pm n}^j$ by relations

$$\sum_{n \in \mathbf{Z}} \psi_n^{\pm,j} z^{-n} = q^{\pm H_0^j} \exp\left(\pm(q - q^{-1}) \sum_{n>0} H_{\pm n}^j z^{\mp n}\right). \tag{II.14}$$

The following relations can be proved by induction:

$$H_n^j = \frac{1}{q - q^{-1}} \sum_{p_1 + 2p_2 + \cdots + np_n = n} \frac{(-1)^{\sum p_i - 1} (\sum p_i - 1)!}{p_1! \cdots p_n!} (q^{-H_0^j} \psi_1^{+,j})^{p_1} \cdots (q^{-H_0^j} \psi_n^{+,j})^{p_n},$$

$$H_{-n}^j = \frac{1}{q^{-1} - q} \sum_{p_1 + 2p_2 + \cdots + np_n = n} \frac{(-1)^{\sum p_i - 1} (\sum p_i - 1)!}{p_1! \cdots p_n!} (q^{H_0^j} \psi_{-1}^{-,j})^{p_1} \cdots (q^{H_0^j} \psi_{-n}^{-,j})^{p_n} \text{(II.15)}$$

The Chevalley generators are related to the Drinfeld generators by the formulae

$$h_i = H_0^i, \quad e_i = X_0^{+,i}, \quad f_i = X_0^{-,i}, \quad i = 1, 2, \cdots, 2N-1,$$

$$h_{2N} = H_0^{2N}, \quad h_0 = c - \sum_{k=1}^{2N-1} H_0^k,$$

$$e_0 = [X_0^{-,2N-1}, [X_0^{-,2N-2}, \cdots, [X_0^{-,3}, [X_0^{-,2}, X_1^{-,1}]_q]_{q^{-1}} \cdots]_q]_{q^{-1}} q^{-\sum_{k=1}^{2N-1} H_0^k},$$

$$f_0 = (-1)^N q^{\sum_{k=1}^{2N-1} H_0^k} [[\cdots [[X_{-1}^{+,1}, X_0^{+,2}]_{q^{-1}} X_0^{+,3}]_q, \cdots, X_0^{+,2N-2}]_{q^{-1}}, X_0^{+,2N-1}]_q \text{(II.16)}$$

The coproduct of the Drinfeld generators is not known in full. However, for our purpose it suffices to derive the "main terms" [19] in the coproduct formulae. We have

Proposition 1 : *For* $m \in \mathbf{Z}_{\geq 0}$, $n \in \mathbf{Z}_{>0}$ *and* $i = 1, 2, \cdots, 2N-1$,

$$\Delta(X_m^{+,i}) = X_m^{+,i} \otimes q^{mc} + q^{H_0^i + 2mc} \otimes X_m^{+,i}$$
$$+ \sum_{k=0}^{m-1} q^{\frac{1}{2}(m+3k)c} \psi_{m-k}^{+,i} \otimes q^{(m-k)c} X_k^{+,i} \quad \text{mod } N_- \otimes N_+^2,$$

$$\Delta(X_{-n}^{+,i}) = X_{-n}^{+,i} \otimes q^{-nc} + q^{-H_0^i} \otimes X_{-n}^{+,i}$$
$$+ \sum_{k=1}^{n-1} q^{\frac{1}{2}(n-k)c} \psi_{k-n}^{-,i} \otimes q^{(k-n)c} X_{-n}^{+,i} \quad \text{mod } N_- \otimes N_+^2,$$

$$\Delta(X_n^{-,i}) = X_n^{-,i} \otimes q^{H_0^i} + q^{nc} \otimes X_n^{-,i}$$
$$+ \sum_{k=1}^{n-1} q^{(n-k)c} X_k^{-,i} \otimes q^{\frac{1}{2}(k-n)c} \psi_{n-k}^{+,i} \quad \text{mod } N_-^2 \otimes N_+,$$

$$\Delta(X_{-m}^{-,i}) = X_{-m}^{-,i} \otimes q^{-H_0^i - 2mc} + q^{-mc} \otimes X_{-m}^{-,i}$$
$$+ \sum_{k=0}^{m-1} q^{(k-m)c} X_{-k}^{-,i} \otimes q^{-\frac{1}{2}(m+3k)c} \psi_{k-m}^{-,i} \quad \text{mod } N_-^2 \otimes N_+,$$

$$\Delta(H_n^i) = H_n^i \otimes q^{\frac{1}{2}nc} + q^{\frac{3}{2}nc} \otimes H_n^i \quad \text{mod } N_- \otimes N_+,$$

$$\Delta(H_{-n}^i) = H_{-n}^i \otimes q^{-\frac{3}{2}nc} + q^{-\frac{1}{2}nc} \otimes H_{-n}^i \quad \text{mod } N_- \otimes N_+, \tag{II.17}$$

where $N_\pm$ *and* $N_\pm^2$ *are the left ideals generated by* $X_l^{\pm,k}$ *and* $X_l^{\pm,k} X_{l'}^{\pm,k'}$, k, $k' = 1, \cdots, 2N-1$; l, $l' \in \mathbf{Z}$, *respectively.*

Remark. (i) We do not write down the formulae for $\Delta(H^{2N}_{\pm n})$ because they are not needed in this paper. $\Delta(H^{2N}_{\pm n})$ can be determined by requiring that Δ preserves the commutation relations (II.13). (ii) Modulo $N_+ \otimes N_- + N_- \otimes N_+$, the elements $\psi^{\pm,i}_{\pm n}$ $(n \geq 0)$ are group-like:

$$\Delta(\psi^{+,i}_n) = \sum_{k=0}^{n} q^{\frac{3}{2}kc} \psi^{+,i}_{n-k} \otimes q^{\frac{1}{2}(n-k)c} \psi^{+,i}_k,$$

$$\Delta(\psi^{-,i}_{-n}) = \sum_{k=0}^{n} q^{-\frac{1}{2}kc} \psi^{-,i}_{k-n} \otimes q^{\frac{3}{2}(k-n)c} \psi^{-,i}_{-k}. \tag{II.18}$$

Define the Drinfeld currents or generating functions,

$$X^{\pm,i}(z) = \sum_{n \in \mathbf{Z}} X^{\pm,i}_n z^{-n-1}, \qquad \psi^{\pm,j}(z) = \sum_{n \in \mathbf{Z}} \psi^{\pm,j}_n z^{-n} \tag{II.19}$$

In terms of these currents, (II.13) read

$$\psi^{\pm,j}(z)\psi^{\pm,j'}(w) = \psi^{\pm,j'}(w)\psi^{\pm,j}(z),$$

$$\psi^{+,j}(z)\psi^{-,j'}(w) = \frac{(z - wq^{c+a_{jj'}})(z - wq^{-c-a_{jj'}})}{(z - wq^{c-a_{jj'}})(z - wq^{-c+a_{jj'}})} \psi^{-,j'}(w)\psi^{+,j}(z),$$

$$\psi^{+,j}(z)X^{\pm,i}(w) = q^{\pm a_{ij}} \frac{z - wq^{\mp\frac{c}{2}\mp a_{ij}}}{z - wq^{\mp\frac{c}{2}\pm a_{ij}}} X^{\pm,i}(w)\psi^{+,j}(z),$$

$$\psi^{-,j}(z)X^{\pm,i}(w) = q^{\pm a_{ij}} \frac{z - wq^{\pm\frac{c}{2}\mp a_{ij}}}{z - wq^{\pm\frac{c}{2}\pm a_{ij}}} X^{\pm,i}(w)\psi^{-,j}(z),$$

$$[X^{+,i}(z), X^{-,i'}(w)] = \frac{\delta_{ii'}}{(q - q^{-1})zw} \left(\delta(\frac{w}{z}q^c)\psi^{+,i}(wq^{\frac{c}{2}}) - \delta(\frac{w}{z}q^{-c})\psi^{-,i}(wq^{-\frac{c}{2}}) \right),$$

$$X^{\pm,i}(z)X^{\pm,i'}(w) + X^{\pm,i'}(w)X^{\pm,i}(z) = 0, \quad \text{for } a_{ii'} = 0,$$

$$(z - wq^{\pm a_{ii'}})X^{\pm,i}(z)X^{\pm,i'}(w) + (zq^{\pm a_{ii'}} - w)X^{\pm,i'}(w)X^{\pm,i}(z) = 0,$$

$$\{[[X^{\pm,l}(z_1), X^{\pm,l-1}(z)]_{q^{(-1)^l}}, [X^{\pm,l}(z_2), X^{\pm,l+1}(w)]_{q^{(-1)^{l+1}}}]\}$$

$$+ \{z_1 \leftrightarrow z_2\} = 0, \quad l = 2, 3, \cdots, 2N - 2. \tag{II.20}$$

These current commutation relations can be derived from the super version [24, 25] of the RS algebra [26] by means of the Gauss decomposition technique of Ding and Frenkel [27].

III   Level-Zero Representation

We consider the evaluation representation V_z of $U_q[gl(\widehat{N|N})]$, where V is an $2N$-dimensional graded vector space with basis vectors $\{v_1, v_2, \cdots, v_{2N}\}$. The $\mathbf{Z}_2$-grading of the basis vectors is chosen to be $[v_j] = \frac{(-1)^j+1}{2}$. Let $e_{j,j'}$ be the $2N \times 2N$ matrix satisfying $(e_{j,j'})_{kk'} = \delta_{jk}\delta_{j'k'}$ or equivalently $e_{i,j}v_k = \delta_{jk}v_i$, (which implies that for any operator A its matrix elements $A_{j,i}$ are defined by $Av_i = A_{j,i}v_j$). In the homogeneous gradation,

the Chevalley generators on V_z are represented by

$$e_i = e_{i,i+1}, \quad f_i = (-1)^{i+1} e_{i+1,i}, \quad i = 1, 2, \cdots, 2N-1,$$

$$h_i = (-1)^{i+1}(e_{i,i} + e_{i+1,i+1}), \quad h_{2N} = \sum_{k=1}^{2N}(-1)^{k+1} e_{k,k},$$

$$e_0 = z e_{2N,1}, \quad f_0 = -z^{-1} e_{1,2N}, \quad h_0 = -e_{1,1} - e_{2N,2N}. \tag{III.1}$$

Let V^{*S} be the left dual module of V, defined by

$$(a \cdot v^*)(v) = (-1)^{[a][v^*]} v^*(S(a)v), \quad \forall a \in U_q[gl(\widehat{N|N})], \ v \in V, \ v^* \in V^*. \tag{III.2}$$

Namely, the representations on V^{*S} are given by

$$\pi_{V^{*S}}(a) = \pi_V(S(a))^{st}, \quad \forall a \in U_q[gl(\widehat{N|N})], \tag{III.3}$$

where st denotes the supertansposition defined by $(A_{i,j})^{st} = (-1)^{[j]([i]+[j])} A_{j,i}$. Note that in general $((A_{i,j})^{st}))^{st} = (-1)^{[A]} A_{i,j} \neq A_{i,j}$. Let V_z^{*S} be the $2N$-dimensional evaluation module corresponding to V^{*S}. On V_z^{*S}, the Chevalley generators are represented by

$$e_i = -(-1)^i q^{(-1)^i} e_{i+1,i}, \quad f_i = -q^{(-1)^{i+1}} e_{i,i+1}, \quad i = 1, 2, \cdots, 2N-1,$$

$$h_i = (-1)^i (e_{i,i} + e_{i+1,i+1}), \quad h_{2N} = \sum_{k=1}^{2N}(-1)^k e_{k,k},$$

$$e_0 = zq e_{1,2N}, \quad f_0 = z^{-1} q^{-1} e_{2N,1}, \quad h_0 = e_{1,1} + e_{2N,2N}. \tag{III.4}$$

Proposition 2 : *The Drinfeld generators are represented on V_z by*

$$H_m^i = (-1)^{i+1} \frac{[m]_q}{m} q^{(-1)^i m} (q^{x_i} z)^m (e_{i,i} + e_{i+1,i+1}),$$

$$H_m^{2N} = z^m \frac{[2m]_q}{m} \left[-q^m \sum_{l=1}^{N} e_{2l,2l} \right.$$

$$\left. + \sum_{l=1}^{N} (y + (l-1)(1-q^m)) (e_{2l-1,2l-1} + e_{2l,2l}) \right],$$

$$H_0^i = (-1)^{i+1}(e_{i,i} + e_{i+1,i+1}), \quad H_0^{2N} = \sum_{k=1}^{2N}(-1)^{k+1} e_{k,k},$$

$$X_m^{+,i} = (q^{x_i} z)^m e_{i,i+1}, \quad X_m^{-,i} = (-1)^{i+1} (q^{x_i} z)^m e_{i+1,i}, \tag{III.5}$$

and on V_z^{*S} by

$$H_m^i = (-1)^i \frac{[m]_q}{m} q^{(-1)^{i+1} m} \left(q^{-x_i} z \right)^m (e_{i,i} + e_{i+1,i+1}),$$

$$H_m^{2N} = -z^m \frac{[2m]_q}{m} \left[-q^{-m} \sum_{l=1}^{N} e_{2l,2l} \right.$$

$$+ \sum_{l=1}^{N} \left(-y^* + (l-1)(1-q^{-m}) \right) \left(e_{2l-1,2l-1} + e_{2l,2l} \right) \Bigg],$$

$$H_0^i = (-1)^i (e_{i,i} + e_{i+1,i+1}), \qquad H_0^{2N} = \sum_{k=1}^{2N} (-1)^k e_{k,k},$$

$$X_m^{+,i} = -(-1)^i q^{(-1)^i} \left(q^{-x_i} z \right)^m e_{i+1,i}, \qquad X_m^{-,i} = -q^{(-1)^{i+1}} \left(q^{-x_i} z \right)^m e_{i,i+1}, \quad \text{(III.6)}$$

where $i = 1, \cdots, 2N-1$, $x_i = \sum_{l=1}^{i} (-1)^{l+1} = \frac{(-1)^{i+1}+1}{2}$ *and* y, y^* *are arbitrary constants.*

IV Free Boson Realization at Level One

We use the notations similar to those in [9, 13]. Let us introduce bosonic oscillators $\{A_n^j, c_n^l, Q_{A^j}, Q_{c^l} | n \in \mathbf{Z}, j = 1, 2, \cdots, 2N, l = 1, 2, \cdots, N\}$ which satisfy the commutation relations

$$[A_n^j, A_m^{j'}] = \delta_{n+m,0} \frac{[a_{jj'} n]_q [n]_q}{n}, \qquad [A_0^j, Q_{A^{j'}}] = a_{jj'},$$

$$[c_n^l, c_m^{l'}] = \delta_{ll'} \delta_{n+m,0} \frac{[n]_q^2}{n}, \qquad [c_0^l, Q_{c^{l'}}] = \delta_{ll'}. \qquad \text{(IV.1)}$$

The remaining commutation relations are zero. Introduce the currents

$$H^j(z; \kappa) = Q_{A^j} + A_0^j \ln z - \sum_{n \neq 0} \frac{A_n^j}{[n]_q} q^{\kappa |n|} z^{-n},$$

$$c^l(z) = Q_{c^l} + c_0^l \ln z - \sum_{n \neq 0} \frac{c_n^l}{[n]_q} z^{-n} \qquad \text{(IV.2)}$$

and set

$$\begin{aligned}
H_\pm^j(z) &= H^j(q^{\pm\frac{1}{2}} z; -\frac{1}{2}) - H^j(q^{\mp\frac{1}{2}} z; \frac{1}{2}) \\
&= \pm (q - q^{-1}) \sum_{n>0} A_{\pm n}^j z^{\mp n} \pm A_0^i \ln q. \qquad \text{(IV.3)}
\end{aligned}$$

We make a basis transformation and express A_n^j and Q_{A^j} in terms of a new set of bosonic oscillators $\{a_n^j, Q_{a^j} | j = 1, 2, \cdots, 2N\}$ as

$$A_n^i = (-1)^{i+1} \left(a_n^i + a_n^{i+1} \right), \qquad A_n^{2N} = \frac{q^n + q^{-n}}{2} \sum_{l=1}^{2N} (-1)^{l+1} a_n^l,$$

$$Q_{A^i} = (-1)^{i+1} \left(Q_{a^i} + Q_{a^{i+1}} \right), \qquad Q_{A^{2N}} = \sum_{l=1}^{2N} (-1)^{l+1} Q_{a^l}, \qquad \text{(IV.4)}$$

where $i = 1, \cdots, 2N-1$ and $\{a_n^j, Q_{a^j}\}$ satisfy the commutation relations

$$[a_n^j, a_m^{j'}] = (-1)^{j+1} \delta_{jj'} \delta_{n+m,0} \frac{[n]_q^2}{n}, \qquad [a_0^j, Q_{a^{j'}}] = (-1)^{j+1} \delta_{jj'}. \qquad \text{(IV.5)}$$

Now we state our main result in this section on the free boson realization of $U_q[gl(\widehat{N|N})]$ at level one.

Theorem 1 : *The Drinfeld generators of $U_q[gl(\widehat{N|N})]$ at level one are realized by the free boson fields as*

$$c = 1,$$

$$\psi^{\pm,j}(z) = e^{H^j_\pm(z)}, \qquad j = 1, 2, \cdots, 2N,$$

$$X^{\pm,i}(z) =: e^{\pm H^i(z;\mp\frac{1}{2})} Y^{\pm,i}(z) : F^{\pm,i}, \qquad i = 1, 2, \cdots, 2N - 1, \qquad \text{(IV.6)}$$

where

$$F^{\pm,2k-1} = \prod_{l=1}^{k-1} e^{\pm\sqrt{-1}\pi a_0^{2l-1}}, \qquad F^{\pm,2k} = \prod_{l=1}^{k} e^{\mp\sqrt{-1}\pi a_0^{2l-1}},$$

$$Y^{+,2k-1}(z) = e^{c^k(z)},$$

$$Y^{-,2k-1}(z) = \frac{1}{z(q - q^{-1})} \left(e^{-c^k(qz)} - e^{-c^k(q^{-1}z)} \right),$$

$$Y^{+,2k}(z) = Y^{-,2k-1}(z) = \frac{1}{z(q - q^{-1})} \left(e^{-c^k(qz)} - e^{-c^k(q^{-1}z)} \right),$$

$$Y^{-,2k}(z) = -Y^{+,2k-1}(z) = -e^{c^k(z)}, \qquad k = 1, 2, \cdots, N. \qquad \text{(IV.7)}$$

Proof. We prove this theorem by checking that they satisfy the defining relations (II.20) of $U_q[gl(\widehat{N|N})]$ with $c = 1$. It is easily seen that the first two relations in (II.20) are true by construction. The third and fourth ones follow from the definition of $X^{\pm,i}(z)$ and the commutativity between a_n^j and c_n^l. So we only need to check the last three relations in (II.20).

We write

$$Z^{\pm,i}(z) =: e^{\pm H^i(z;\mp\frac{1}{2})} : F^{\pm,i}. \qquad \text{(IV.8)}$$

It is easily shown that

$$Z^{+,i}(z)Z^{+,i'}(w) = \begin{cases} : Z^{+,i}(z)Z^{+,i'}(w) : & \text{for } a_{ii'} = 0 \text{ and } i \le i', \\ - : Z^{+,i}(z)Z^{+,i'}(w) : & \text{for } a_{ii'} = 0 \text{ and } i > i', \\ (z - q^{-1}w) : Z^{+,i}(z)Z^{+,i'}(w) : & \text{for } a_{ii'} = 1 \text{ and } i < i', \\ -(z - q^{-1}w) : Z^{+,i}(z)Z^{+,i'}(w) : & \text{for } a_{ii'} = 1 \text{ and } i > i', \\ (z - q^{-1}w)^{-1} : Z^{+,i}(z)Z^{+,i'}(w) : & \text{for } a_{ii'} = -1, \end{cases} \qquad \text{(IV.9)}$$

$$Z^{-,i}(z)Z^{-,i'}(w) = \begin{cases} : Z^{-,i}(z)Z^{-,i'}(w) : & \text{for } a_{ii'} = 0 \text{ and } i \le i', \\ - : Z^{-,i}(z)Z^{-,i'}(w) : & \text{for } a_{ii'} = 0 \text{ and } i > i' \\ (z - qw) : Z^{-,i}(z)Z^{-,i'}(w) : & \text{for } a_{ii'} = 1 \text{ and } i < i', \\ -(z - qw) : Z^{-,i}(z)Z^{-,i'}(w) : & \text{for } a_{ii'} = 1 \text{ and } i > i', \\ (z - qw)^{-1} : Z^{-,i}(z)Z^{-,i'}(w) : & \text{for } a_{ii'} = -1, \end{cases} \qquad \text{(IV.10)}$$

$$Z^{+,i}(z)Z^{-,i'}(w) = \begin{cases} : Z^{+,i}(z)Z^{-,i'}(w): & \text{for } a_{ii'}=0 \text{ and } i \leq i', \\ -: Z^{+,i}(z)Z^{-,i'}(w): & \text{for } a_{ii'}=0 \text{ and } i > i', \\ (z-w)^{-1} : Z^{+,i}(z)Z^{-,i'}(w): & \text{for } a_{ii'}=1 \text{ and } i < i', \text{(IV.11)} \\ -(z-w)^{-1} : Z^{+,i}(z)Z^{-,i'}(w): & \text{for } a_{ii'}=1 \text{ and } i > i', \\ (z-w) : Z^{+,i}(z)Z^{-,i'}(w): & \text{for } a_{ii'}=-1, \end{cases}$$

We have similar formulae for $Z^{+,i'}(w)Z^{+,i}(z)$, $Z^{-,i'}(w)Z^{-,i}(z)$ and $Z^{-,i'}(w)Z^{+,i}(z)$.

We now compute operator products $Y^{+,i}(z)Y^{+,i'}(w)$ and $Y^{+,i}(z)Y^{-,i'}(w)$. It is easily seen from the definition of $Y^{\pm,i}(z)$ that the non-trivial products are those corresponding to $i = i'$ and $a_{ii'} = 1$. Note that $a_{ii'} = 1$ whenever $i = 2k-1$, $i' = 2k$ (or $i = 2k$, $i' = 2k-1$) where $k = 1, 2, \cdots, N-1$. The corresponding operator products are

$$Y^{+,2k-1}(z)Y^{+,2k}(w) = \frac{1}{w(q-q^{-1})}\left(\frac{: e^{c^k(z)}e^{-c^k(qw)}:}{z-qw} - \frac{: e^{c^k(z)}e^{-c^k(q^{-1}w)}:}{z-q^{-1}w}\right),$$

$$Y^{-,2k-1}(z)Y^{-,2k}(w) = -\frac{1}{z(q-q^{-1})}\left(\frac{: e^{-c^k(qz)}e^{c^k(w)}:}{qz-w} - \frac{: e^{-c^k(q^{-1}z)}e^{c^k(w)}:}{q^{-1}z-w}\right),$$

$$Y^{+,2k-1}(z)Y^{-,2k}(w) = -(z-w) : Y^{+,2k-1}(z)Y^{-,2k}(w):$$

$$Y^{+,2k}(z)Y^{-,2k-1}(w) = \frac{1}{zw(q-q^{-1})}\Big(q(z-w) : e^{-c^k(qz)}e^{-c^k(qw)}:$$
$$-q^{-1}(z-w) : e^{-c^k(q^{-1}z)}e^{-c^k(q^{-1}w)}:$$
$$-(qz-q^{-1}w) : e^{-c^k(qz)}e^{-c^k(q^{-1}w)}:$$
$$-(q^{-1}z-qw) : e^{-c^k(q^{-1}z)}e^{-c^k(qw)}:\Big). \tag{IV.12}$$

Since $Y^{+,2k}(z) = Y^{-,2k-1}(z)$ and $Y^{-,2k}(z) = -Y^{+,2k-1}(z)$, the products $Y^{+,i}(z)Y^{+,i}(w)$ and $Y^{\pm,i}(z)Y^{\mp,i}(w)$ can be deduced from (IV.12). For example,

$$Y^{+,2k}(z)Y^{-,2k}(w) = -Y^{+,2k}(z)Y^{+,2k-1}(w)$$
$$= \frac{1}{z(q-q^{-1})}\left(\frac{: e^{-c^k(q^{-1}z)}e^{c^k(w)}:}{q^{-1}z-w} - \frac{: e^{-c^k(qz)}e^{c^k(w)}:}{qz-w}\right). \tag{IV.13}$$

By means of (IV.9), (IV.11), (IV.10) and (IV.12) we can show that the last three relations in (II.20) are satisfied by (IV.7). For instance,

$$[X^{+,2k}(z), X^{-,2k-1}(w)] = -\frac{1}{zw(q-q^{-1})^2} : Z^{+,2k}(z)Z^{-,2k-1}(w): \left(\frac{1}{z-w}+\frac{1}{w-z}\right)$$
$$\times \Big((qz-q^{-1}w) : e^{-c^k(qz)}e^{-c^k(q^{-1}w)}:$$
$$+(q^{-1}z-qw) : e^{-c^k(q^{-1}z)}e^{-c^k(qw)}:\Big)$$
$$= -\frac{1}{z^2w(q-q^{-1})^2} : Z^{+,2k}(z)Z^{-,2k-1}(w):$$
$$\times \delta(\frac{w}{z})\Big((qz-q^{-1}w) : e^{-c^k(qz)}e^{-c^k(q^{-1}w)}:$$
$$+(q^{-1}z-qw) : e^{-c^k(q^{-1}z)}e^{-c^k(qw)}:\Big) = 0. \tag{IV.14}$$

V   Bosonization of Level-One Vertex Operators

In this section, we study the level-one vertex operators [6] of $U_q[gl(\widehat{N|N})]$. Let $V(\lambda)$ be the highest weight $U_q[gl(\widehat{N|N})]$-module with the highest weight λ. Consider the following intertwiners of $U_q[gl(\widehat{N|N})]$-modules [2]:

$$\Phi_\lambda^{\mu V}(z) \quad : \quad V(\lambda) \longrightarrow V(\mu) \otimes V_z, \tag{V.1}$$

$$\Phi_\lambda^{\mu V^*}(z) \quad : \quad V(\lambda) \longrightarrow V(\mu) \otimes V_z^{*S}, \tag{V.2}$$

$$\Psi_\lambda^{V\mu}(z) \quad : \quad V(\lambda) \longrightarrow V_z \otimes V(\mu), \tag{V.3}$$

$$\Psi_\lambda^{V^*\mu}(z) \quad : \quad V(\lambda) \longrightarrow V_z^{*S} \otimes V(\mu). \tag{V.4}$$

They are intertwiners in the sense that for any $x \in U_q[gl(\widehat{N|N})]$

$$\Xi(z) \cdot x = \Delta(x) \cdot \Xi(z), \quad \Xi(z) = \Phi_\lambda^{\mu V}(z),\ \Phi_\lambda^{\mu V^*}(z),\ \Psi_\lambda^{V\mu}(z),\ \Psi_\lambda^{V^*\mu}(z). \tag{V.5}$$

These intertwiners are even operators, that is their gradings are $[\Phi_\lambda^{\mu V}(z)] = [\Phi_\lambda^{\mu V^*}(z)] = [\Psi_\lambda^{V\mu}(z)] = [\Psi_\lambda^{V^*\mu}(z)] = 0$. According to [2], $\Phi_\lambda^{\mu V}(z)$ $\left(\Phi_\lambda^{\mu V^*}(z)\right)$ is called type I (dual) vertex operator and $\Psi_\lambda^{V\mu}(z)$ $\left(\Psi_\lambda^{V^*\mu}(z)\right)$ type II (dual) vertex operator.

We expand the vertex operators as a formal series [2]

$$\Phi_\lambda^{\mu V}(z) = \sum_{j=1}^{2N} \Phi_{\lambda,j}^{\mu V}(z) \otimes v_j, \quad \Phi_\lambda^{\mu V^*}(z) = \sum_{j=1}^{2N} \Phi_{\lambda,j}^{\mu V^*}(z) \otimes v_j^*,$$

$$\Psi_\lambda^{V\mu}(z) = \sum_{j=1}^{2N} v_j \otimes \Psi_{\lambda,j}^{V\mu}(z), \quad \Psi_\lambda^{V^*\mu}(z) = \sum_{j=1}^{2N} v_j^* \otimes \Psi_{\lambda,j}^{V^*\mu}(z). \tag{V.6}$$

Then the intertwining property (V.5) reads in terms of components

$$\sum \Phi_{\lambda,j}^{\mu V}(z)\, x \otimes v_j (-1)^{[v_j][x]} = \sum x_{(1)} \Phi_{\lambda,j}^{\mu V}(z) \otimes x_{(2)} v_j (-1)^{[v_j][x_{(2)}]},$$

$$\sum \Phi_{\lambda,j}^{\mu V^*}(z)\, x \otimes v_j^* (-1)^{[v_j^*][x]} = \sum x_{(1)} \Phi_{\lambda,j}^{\mu V^*}(z) \otimes x_{(2)} v_j^* (-1)^{[v_j^*][x_{(2)}]},$$

$$\sum v_j \otimes \Psi_{\lambda,j}^{V\mu}(z) x = \sum x_{(1)} v_j \otimes x_{(2)} \Psi_{\lambda,j}^{V\mu}(z)(-1)^{[v_j][x_{(2)}]},$$

$$\sum v_j^* \otimes \Psi_{\lambda,j}^{V^*\mu}(z) x = \sum x_{(1)} v_j^* \otimes x_{(2)} \Psi_{\lambda,j}^{V^*\mu}(z)(-1)^{[v_j^*][x_{(2)}]}, \tag{V.7}$$

where we have used the notation $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$ and the fact that the vertex operators are even which implies $[\Phi_{\lambda,j}^{\mu V}(z)] = [\Phi_{\lambda,j}^{\mu V^*}(z)] = [\Psi_{\lambda,j}^{V\mu}(z)] = [\Psi_{\lambda,j}^{V^*\mu}(z)] = [v_j] = \frac{(-1)^j+1}{2}$.

Introduce the even operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$,

$$\phi(z) = \sum_{j=1}^{2N} \phi_j(z) \otimes v_j, \quad \phi^*(z) = \sum_{j=1}^{2N} \phi_j^*(z) \otimes v_j^*,$$

$$\psi(z) = \sum_{j=1}^{2N} v_j \otimes \psi_j(z), \quad \psi^*(z) = \sum_{j=1}^{2N} v_j^* \otimes \psi_j^*(z). \tag{V.8}$$

The grading of the components is given by $[\phi_j(z)] = [\phi_j^*(z)] = [\psi_j(z)] = [\psi_j^*(z)] = \frac{(-1)^{j+1}}{2}$.
Now we state

Proposition 3 : *Assume that the operators* $\phi(z)$, $\phi^*(z)$, $\psi(z)$, $\psi^*(z)$ *satisfy the inter-twining relations (V.7). Then the operators* $\phi(z)$ *and* $\psi(z)$ *with respect to* V_z *are determined by the components* $\phi_{2N}(z)$ *and* $\psi_1(z)$, *respectively. With respect to* V_z^{*S}, *the operators* $\phi^*(z)$ *and* $\psi^*(z)$ *are determined by* $\phi_1^*(z)$ *and* $\psi_{2N}^*(z)$, *respectively. More explicitly, we have for* $l = 1, 2, \cdots, 2N - 1$,

$$(-1)^l \, \phi_l(z) = [\phi_{l+1}(z), f_l]_{q^{(-1)^l}},$$
$$[\phi_l(z), f_l]_{q^{(-1)^l}} = 0,$$
$$[\phi_k(z), f_l] = 0, \quad k \neq l, \, l+1. \tag{V.9}$$

$$q^{(-1)^{l+1}} \, \phi_{l+1}^*(z) = [\phi_l^*(z), f_l]_{q^{(-1)^{l+1}}},$$
$$[\phi_{l+1}^*(z), f_l]_{q^{(-1)^{l+1}}} = 0,$$
$$[\phi_k^*(z), f_l] = 0, \quad k \neq l, \, l+1. \tag{V.10}$$

$$\psi_{l+1}(z) = [\psi_l(z), e_l]_{q^{(-1)^{l+1}}},$$
$$[\psi_{l+1}(z), e_l]_{q^{(-1)^{l+1}}} = 0,$$
$$[\psi_k(z), e_l] = 0, \quad k \neq l, \, l+1. \tag{V.11}$$

$$(-1)^{l+1} q^{(-1)^l} \, \psi_l^*(z) = [\psi_{l+1}^*(z), e_l]_{q^{(-1)^l}},$$
$$[\psi_l^*(z), e_l]_{q^{(-1)^l}} = 0,$$
$$[\psi_k^*(z), e_l] = 0, \quad k \neq l, \, l+1. \tag{V.12}$$

Next we determine the relations of the components $\phi_{2N}(z)$, $\phi_1^*(z)$, $\psi_1(z)$, $\psi_{2N}^*(z)$ and the Drinfeld generators. By means of proposition 1 and the intertwining relations, we have

Proposition 4 : *For* $\phi(z)$ *associated with* V_z,

$$[\phi_{2N}(z), X^{+,i}(w)] = 0,$$
$$q^{h_i} \phi_{2N}(z) q^{-h_i} = q^{-\delta_{i,2N-1}} \phi_{2N}(z),$$
$$[H_n^i, \phi_{2N}(z)] = -\delta_{i,2N-1} q^{\frac{3}{2}n} \frac{[n]_q}{n} z^n \phi_{2N}(z),$$
$$[H_{-n}^i, \phi_{2N}(z)] = -\delta_{i,2N-1} q^{-\frac{1}{2}n} \frac{[n]_q}{n} z^{-n} \phi_{2N}(z); \tag{V.13}$$

for $\phi^*(z)$ associated with V_z^*,

$$[\phi_1^*(z), X^{+,i}(w)] = 0,$$

$$q^{h_i}\phi_1^*(z)q^{-h_i} = q^{\delta_{i,1}}\phi_1^*(z),$$

$$[H_n^i, \phi_1^*(z)] = \delta_{i,1}q^{\frac{3}{2}n}\frac{[n]_q}{n}z^n\phi_1^*(z),$$

$$[H_{-n}^i, \phi_1^*(z)] = \delta_{i,1}q^{-\frac{1}{2}n}\frac{[n]_q}{n}z^{-n}\phi_1^*(z); \tag{V.14}$$

for $\psi(z)$ associated with V_z,

$$[\psi_1(z), X^{-,i}(w)] = 0,$$

$$q^{h_i}\psi_1(z)q^{-h_i} = q^{-\delta_{i,1}}\psi_1(z),$$

$$[H_n^i, \psi_1(z)] = -\delta_{i,1}q^{\frac{1}{2}n}\frac{[n]_q}{n}z^n\psi_1(z),$$

$$[H_{-n}^i, \psi_1(z)] = -\delta_{i,1}q^{-\frac{3}{2}n}\frac{[n]_q}{n}z^{-n}\psi_1(z); \tag{V.15}$$

and for $\psi^*(z)$ associated with V_z^*,

$$[\psi_{2N}^*(z), X^{-,i}(w)] = 0,$$

$$q^{h_i}\psi_{2N}^*(z)q^{-h_i} = q^{\delta_{i,2N-1}}\psi_{2N}^*(z),$$

$$[H_n^i, \psi_{2N}^*(z)] = \delta_{i,2N-1}q^{\frac{1}{2}n}\frac{[n]_q}{n}z^n\psi_{2N}^*(z),$$

$$[H_{-n}^i, \psi_{2N}^*(z)] = \delta_{i,2N-1}q^{-\frac{3}{2}n}\frac{[n]_q}{n}z^{-n}\psi_{2N}^*(z). \tag{V.16}$$

In order to obtain bosonized expressions of the vertex operators, we introduce the following combinations of the Drinfeld generators:

$$A_n^{*i} = \sum_{l=1}^{2N-1} a_{il}^{-1}A_n^l + \frac{2}{q^n + q^{-n}}a_{i,2N}^{-1}A_n^{2N},$$

$$A_0^{*i} = \sum_{l=1}^{2N} a_{il}^{-1}A_0^l, \quad Q_{A^i}^* = \sum_{l=1}^{2N} a_{il}^{-1}Q_{A^l}, \quad i = 1, 2, \cdots, 2N - 1, \tag{V.17}$$

which satisfy the relations

$$[A_n^{*i}, A_m^{i'}] = \delta_{ii'}\delta_{n+m,0}\frac{[n]_q^2}{n},$$

$$[A_n^{*i}, A_m^{*i'}] = a_{ii'}^{-1}\delta_{n+m,0}\frac{[n]_q^2}{n},$$

$$[A_0^{*i}, Q_{A^{i'}}] = \delta_{ii'}, \quad [A_0^i, Q_{A^{i'}}^*] = \delta_{ii'},$$

$$[A_0^{*i}, Q_{A^{i'}}^*] = a_{ii'}^{-1}, \quad i, i' = 1, 2, \cdots, 2N - 1. \tag{V.18}$$

Introduce the currents,

$$H^{*,j}(z;\kappa) = Q_{A^j}^* + A_0^{*j}\ln z - \sum_{n\neq 0}\frac{A_n^{*j}}{[n]_q}q^{k|n|}z^{-n}. \tag{V.19}$$

Now we state our main theorem in this section on the bosonic realization of the operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$ at level one. Thanks to the previous propositions, we only need to determine one component for each operator and the other components are represented by the integral of the currents.

Theorem 2 : *The components $\phi_{2N}(z)$, $\phi_1^*(z)$, $\psi_1(z)$ and $\psi_{2N}^*(z)$ can be realized explicitly as follows:*

$$\phi_{2N}(z) = \ :e^{-H^{*,2N-1}(qz;\frac{1}{2})}e^{c^N(qz)}: \ e^{-\sqrt{-1}\pi a_0^1}\prod_{l=1}^{N-1}e^{-\sqrt{-1}\pi\frac{2N+l}{2N}a_0^{2l+1}},$$

$$\phi_1^*(z) = \ :e^{H^{*,1}(qz;\frac{1}{2})}: \ \prod_{l=1}^{N-1}e^{\sqrt{-1}\pi\frac{2N-l}{2N}a_0^{2l+1}},$$

$$\psi_1(z) = \ :e^{-H^{*,1}(qz;-\frac{1}{2})}: \ \prod_{l=1}^{N-1}e^{\sqrt{-1}\pi\frac{2N-l}{2N}a_0^{2l+1}},$$

$$\psi_{2N}^*(z) = \ \frac{1}{z(q-q^{-1})}:e^{H^{*,2N-1}(qz;-\frac{1}{2})}\left(e^{-c^N(q^2z)}-e^{-c^N(z)}\right):$$

$$\times\ e^{-\sqrt{-1}\pi a_0^1}\prod_{l=1}^{N-1}e^{-\sqrt{-1}\pi\frac{2N+l}{2N}a_0^{2l+1}}. \tag{V.20}$$

Proof. This theorem is proved by checking that the construction satisfies all the intertwining relations.

Remark. The following inverse elements of the extended Cartan matrix are needed to determine the cocycle factors appearing in above theorem:

$$a_{2N-1,2l}^{-1} = a_{2N-1,2l+1}^{-1} = -\frac{l}{N}, \quad l = 1,2,\cdots,N-1,$$

$$a_{2N-1,1}^{-1} = 0, \quad a_{2N-1,2N}^{-1} = \frac{1}{2N},$$

$$a_{1,2l-1}^{-1} = a_{1,2l}^{-1} = \frac{N-l}{N}, \quad l = 1,2,\cdots,N-1,$$

$$a_{1,2N-1}^{-1} = 0, \quad a_{1,2N}^{-1} = \frac{1}{2N}. \tag{V.21}$$

We are now in a position to state the following result:

Proposition 5 : *The vertex operators $\Phi_\lambda^{\mu V}(z)$, $\Phi_\lambda^{\mu V^*}(z)$, $\Psi_\lambda^{V\mu}(z)$ and $\Psi_\lambda^{V^*\mu}(z)$, if they exist, satisfy the same relations as the operators $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$, respectively.*

This proposition follows immediately from the fact that the formers and the latters obey the same intertwining properties. Identifying $\Phi_\lambda^{\mu V}(z)$, $\Phi_\lambda^{\mu V^*}(z)$, $\Psi_\lambda^{V\mu}(z)$ and $\Psi_\lambda^{V^*\mu}(z)$ with $\phi(z)$, $\phi^*(z)$, $\psi(z)$ and $\psi^*(z)$, respectively, then the bosonic realization of the vertex operators is easily seen to be given by propositions 3, 4 and theorem 2.

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